ON THE FOUNDATION AND TECHNIC OF ARITHMETIC.*

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THE TWO DIRECT OPERATIONS, ADDITION AND MULTIPLICATION.

Notation.

THE symbolic representation of numbers and ways of combining numbers comes under the head of what is called *notation.*

The natural numbers, as shown in the primitive numeral pictures 1, 11, 111, 1111, begin with a single unit, and, cardinally considered, are changed to the next always by taking another single unit.

The Symbol =.

A number, an integer, is said to be *equal* to, or the same as, a number otherwise expressed, when their units being counted come to the same finger, the same numeral word. The symbol =, read *equals,* is called the sign of equality, and takes the part of verb in this symbolic language. It was invented by an Englishman, Robert Recorde, replacing in his algebra, *The Whetstone of Witte,*† the sign z used for equality in his arithmetic, *The Grounde of Artes,* 1540. Equality is a relation reflexive, symmetric, invertible. Equality is a mutual relation of its two members. If $x=y$, then $y=x$. Equality is a transitive relation. If $x=y$ and $y=z$, then $x=z$. A symbolic sentence using this verb is called an equality.

Ordinarily, $x=y$ means that x and y denote the same number in the natural scale. Formally, $x=y$ means that either can at will be substituted for the other anywhere.

* Continuation of an article begun in the February *Open Court.*

† London (no date, preface 1557).

Inequality.

When the process of counting the units of one number simultaneously one-to-one with units of a second number ends because no unit of the second number remains uncounted, but the units of the first number are not all counted, then the first number is said to contain more units than the second number, and the second number is said to contain less units than the first.

If a number contains more units than a second, it is called *greater* than this second, which is called the *lesser*. By successively incorporating single units with the lesser of two primitive numbers we can make the greater.

Thomas Harriot (1560–1621), tutor to Sir Walter Raleigh and one of "the three magi of the Earl of Northumberland," devised the symbol $>$, published 1631, read "is greater than," and called the sign of inequality. Inequality is a sensed relation. Turned thus $<$ its symbol is read "is less than." Inequality in the same sense is transitive. If $x > y$ and $y > z$, then $x > z$.

Since the result of counting is independent of the order of the individuals counted, therefore of two unequal natural numbers the one once found greater is always the greater. Without knowing the number n, we can write "either $n > 5$, or $n = 5$, or $n < 5$." Any number which succeeds another in the natural scale is greater than this other. Ordinally, $x < y$ means that x precedes y in the scale.

Parentheses.

When we can get a third number from two given numbers by a definite operation, the two given numbers joined by the sign for the operation and enclosed in parentheses may be taken to mean the result of that combination. The result can now be again combined with another given number, and so we may get combinations of several numbers though each operation is performed only with two.

Parentheses indicate that neither of the two numbers enclosed, but only the number produced by their combination, is related to anything outside the parentheses.

Parentheses (first used by the Flemish geometer Albert Girard in 1629) may without ambiguity be omitted:

First, When of two operations of like rank the preceding (going from left to right) is to be first carried out;

Second, When of two operations of unlike rank the higher is the first to be carried out.

Expressions.

The representation of one number by others with symbols of combination and operation is called an expression. By enclosing it in parentheses, any expression however complex in any way representing a number, may be operated upon as if it were a single symbol of that number. If an expression already involving parentheses is enclosed in parentheses, each pair, to distinguish it, can be made different in size or shape. The three most usual forms are the parenthesis (, the bracket [, and the brace {. In translating the expression into English, (should be called first parenthesis, and) second parenthesis; [first bracket,] second bracket; { first brace, } second brace.

Substitution.

No change of resulting value is made in any expression by substituting for any number its equal however expressed. From this it follows that two numbers each equal to a third are equal to one another. This process, putting one expression for another, substitution, is a primitive yet most important proceeding. A single symbol may be substituted for any expression whatever.

Permutation consists in a simultaneous carrying out of mutual substitution, interchange. Thus a and b in an expression, as abc, are permuted when they are interchanged, giving bac. More than two symbols are permuted when each is replaced by one of the others, as in abc giving bca or cab.

Addition.

Suppose we have two natural numbers written in their primitive form, as 111 and 1111; if we write all these units in one row we indicate another natural number; and the process of getting from two numbers the number belonging to the group formed by putting together their groups to make a single group is called *addition*. This operation of incorporating other units into the preceding diagram is indicated by a symbol first met in print in the arithmetic by John Widman (Leipsic, 1489), a little Maltese cross, +, read plus.

If one artificial individual be combined with another to give a new artificial individual in which each unit of the components appears retaining its natural independence and natural individuality, while the artificial individuality of the two components vanishes, the number of the new artificial individual is called the *sum* of the numbers of the two components, and is said to be obtained by *adding* these two numbers (the terms or summands). The sum of two numbers, two terms, is the numeric attribute of the total system constituted of two partial systems to which the two terms respectively pertain.

In the child as in the savage, the number idea is not dissociated from the group it characterizes. But education should help on the stage where the number exists as an independent concept, say the number 5 with its own characteristics, its own life. Therefore we have number-science, pure arithmetic. So though it might perhaps be argued that there is only one number 5, yet we may properly speak of combining 5 with 5 so as to retain the units unaffected while the fiveness vanishes in the compound, the sum, 10.

Addition is a taking together of the units of two numbers to constitute the units of a third, their sum. This may be attained by a repetition of the operation of forming a new number from an old by taking with it one more unit; thus $3+2=3+1+1$.

If given numbers are written as groups of units, e. g. (*exempli gratia*), $2=1+1$, $3=1+1+1$, the result of adding is obtained by writing together these rows of units, e. g., $2+3=(1+1)+(1+1+1)$ $=1+1+1+1+1=5$.

Since cardinal number is independent of the order of counting, therefore in any natural number expressed in its primitive form, as 1111, the permutation of any pair of units produces neither apparent nor real change. The units of numeration are completely interchangeable. Therefore we may say adding numbers is finding one number which contains in itself as many units as the given numbers taken together.

In defining addition, we need make no mention of the order in which the given numbers are taken together to make the sum. A sum is independent of the order of its parts or terms. This is an immediate consequence of the theorem of the invariance of the number of a set. For a change in the order of the parts added is only a change in the order of the units, which change is without influence when all are counted together.

To write in symbols, in the universal language of mathematics, that addition is an operation unaffected by permutation or the order of the parts added, though applied to any numbers whatsoever, we cannot use numerals, since numerals are always absolutely definite, particular. If, following Vieta's book of 1591, we use letters as general symbols to denote numbers left otherwise indefinite, we may write a to represent the first number not only in the sum $2+3$, but in the sum $4+1$ and in the sum of any two numbers. Taking b for a second number, the symbolic sentence $a+b=b+a$ is a statement about all numbers whatsoever. It says, addition is a *commutative* operation.

The words commutative and *distributive* were used for the first time by F. J. Servois in 1813.

The previous grouping of the parts added has no effect upon the sum. Brackets occurring in an indicated sum may be omitted as not affecting the result. The general statement or formula $(a+b)+c=a+(b+c)$ says, addition is an *associative* operation, an operation having associative freedom.

Rowan Hamilton in 1844 first explicitly stated and named the associative law. For addition it follows from the theorem of the invariance of the number of a group.

Formulas.

Equalities having to do only with the very nature of the operations involved, and not at all with the particular numbers used are called *formulas.*

A formula is characterized by the fact that for any letter in it any number whatsoever may be substituted without destroying the equality or restricting the values of any other letter. In a formula a letter as symbol for any number may be replaced not only by any digital number, but also by any other symbol for a number whether simple or compound, in the last case bracketed. Thus $a+b=b+a$ gives $(a+c)+b=b+(a+c)$. So from a formula we can get an indefinite number of formulas and special numerical equations.

Each side or member of a formula expresses a method of reckoning a number, and the formula says that both reckonings produce the same result. A formula translated from symbols into words gives a rule. As equality is a mutual relation always invertible, a formula will usually give two rules, since its second member may be read first.

By definition, from the inequality $a>b$ we know that a could be obtained by adding units to b. Calling this unknown group of units n, we have $a=b+n$.

Inversely, if $a=b+n$ then $a>b$, that is a sum of finite natural numbers is always greater than one of its parts. A sum increases if either of its parts increases.

Ordinal Addition.

Addition may also be defined and its properties established from the ordinal view-point.

Start from the natural scale. To add 1 to the number x is to replace x by the next following ordinal. So if we know x, we know $x+1$.

When we have defined adding some particular number a to x, when we have defined the operation, $x+a$, the operation $x+(a+1)$ shall be defined by the formula $(1) \ldots . x+(a+1)=(x+a)+1$. We shall know then what $x+(a+1)$ is when we know what $x+a$ is, and as we have, to start with, defined what $x+1$ is, we thus have successively and "by recurrence" the operations $x+2$, $x+3$, etc.

The sum $a+b$ is thus defined ordinally as the bth term after the ath.

It serves to represent conventionally a new number univocally deduced by a definite given procedure from the numbers summed or added together.

Properties of Addition.

Associativity: $a+(b+c)=(a+b)+c$.

This theorem is by definition true for $c=1$, since, by formula (1), $a+(b+1)=(a+b)+1$. Now supposing the theorem true for $c=y$, it will be true for $c=y+1$. For supposing

$$(a+b)+y=a+(b+y),$$

it follows that

$$(2) \ldots . [(a+b)+y]+1=[a+(b+y)]+1,$$

which is only adding one to the same number, to equal numbers.

Now by definition (1), the first member of this equation (2)

$$[(a+b)+y]+1=(a+b)+(y+1) \ldots . (3),$$

as we recognize that it should be, since y is the number preceding $y+1$.

But by the same formula (1), read backward, the second member of equation (2)

$$[a+(b+y)]+1=a+[(b+y)+1] \ldots . (4),$$

as we see it should be, since $b+y$ is the number preceding $b+y+1$. But again by (1), the second member of (4),

$$a+[(b+y)+1]=a+[b+(y+1)] \ldots . (5)$$

Therefore [by (5), (4) and (3)], (2) may be written,

$$a+[b+(y+1)]=(a+b)+(y+1).$$

Hence the theorem is true for $c=y+1$.

Being true for $c=1$. we thus see successively that so it is for $c=2$, for $c=3$, etc.

Commutativity: $1^{\circ} \ldots . \; a+1=1+a$.

This theorem is identically true for $a=1$.

Now we can verify that if it is true for $a=y$ it will be true for $a=y+1$; for then

$$(y+1)+1=(1+y)+1=1+(y+1)$$

by associativity. But it is true for $a=1$, therefore it will be true for $a=2$, for $a=3$, etc.

$2^{\circ}\ldots\ a+b=b+a.$

This has just been demonstrated for $b=1$; it can be verified that if it is true for $b=x$, it will be true for $b=x+1$. For, if true for $b=x$, then we have by hypothesis $a+x=x+a$; whence, by formula (1), by 1° and associativity, $a+(x+1)=(a+x)+1=(x+a)+1=x+(a+1)$ $=x+(1+a)=(x+1)+a$.

The proposition is therefore established by recurrence.

Multiplication.

Sums in which all the parts are equal frequently occur. Such additions are often laborious and liable to error. But such a sum is *determined* if we know one of the equal parts and the number of parts. The operation of combining these two numbers to get the result is called *multiplication*; the result is then called the *product.* The part repeated is called the *multiplicand,* and the number which indicates how often it occurs is called the *multiplier.* Multiplicand and multiplier are each *factors* of the product. Such a product is a *multiple* of each of its factors. In forming such a product, the multiplicand is taken once as summand for each unit in the multiplier. More generally, *a product is the number related to the multiplicand as the unit to the multiplier.*

Following Wm. Oughtred (1631), we use the sign $\times$ to denote multiplication, writing it before the multiplier but after the multiplicand. Thus 1×10, read one multiplied by ten, or simply one by ten, stands for the product of the multiplication of 1 by 10, which by definition equals 10. The multiplication sign may be omitted when the product cannot reasonably be confounded with anything else, thus $1a$ means $1\times a$, read one by a, which by definition equals a.

From our definition also $a\times 1$, that is a multiplied by 1, must equal a.

Commutativity. Multiplication of a number by a number is commutative.

Multiplier and multiplicand may be interchanged without altering the product.

1 1 1 1 1 For if we have a rectangular array of a rows each
1 1 1 1 1 containing b units, it is also b columns each contain-
1 1 1 1 1 ing a units.

Therefore $b\times a=a\times b$.

Taking apposition to mean successive multiplication, for example, $abcde = \{[(ab)c]d\}\ e$, calling the numbers involved *factors,* and the result their *product,* we may prove that commutative freedom extends to any or all factors in any product.

For changing the order of a pair of factors which are next one another does not alter the product. $abcd = acbd$.

$a\ a\ a\ a\ a$ For c rows of a's, each row containing b of them,
$a\ a\ a\ a\ a$ is b columns of a's each containing c of them. So
$a\ a\ a\ a\ a$ c groups of ab units comes to the same number as b groups of ac units.

This reasoning holds no matter how many factors come before or after the interchanged pair. For example

$$abcdefg = abc\ ed\ fg,$$

since in this case the product abc simply takes the place which the number a had before. And e rows with d times abc in each row come to the same number as d columns with e times abc in each column. It remains only to multiply this number successively by whatever factors stand to the right of the interchanged pair.

It follows therefore that no matter how many numbers are multiplied together, we may interchange the places of any two of them which are adjacent without altering the product. But by repeated interchanges of adjacent pairs we may produce any alteration we choose in the order of the factors.

This extends the commutative law of freedom to all the factors in any product.

Associativity. To show with equal generality that multiplication is associative, we have only to prove that in any product any group of the successive factors may be replaced by their product.

$$abcdefgh = abc(def)gh.$$

By the commutative law we may arrange the factors so that this group comes first. Thus $abcdefgh = def\ abc\ gh$.

But now the product of this group is made in carrying out the multiplication according to definition. Therefore

$$abcdefgh = def\ abc\ gh = (def)\ abc\ gh.$$

Considering this bracketed product now as a single factor of the whole product, it can, by the commutative law, be brought into any position among the other factors, for example, back into the old place; so $abcdefgh = def\ abc\ gh = (def)\ abc\ gh = abc\ (def)\ gh$.

Distributivity. Multiplication combines with addition according to what is called the *distributive* law.

Instead of multiplying a sum and a number we may multiply each part of the sum with the number and add these partial products.

$$a\ (b+c) = (b+c)\ a = ab + ac.$$

$$4 \times 5 = 4(2+3) = (2+3)4 = 2 \times 4 + 3 \times 4 = 5 \times 4.$$

. . . . . Four by five equals five by four, and four rows of
. . . . . $(2+3)$ units may be counted as four rows of two
. . . . . units together with 4 rows of 3 units.
. . . . . As the sum of two numbers is a number, we may
substitute $(a+b)$ for b in the formula $(b+c)d = bd+cd$, which thus gives $[(a+b)+c]d = (a+b)d+cd = ad+bd+cd$. So the distributive law extends to the sum of however many numbers or terms.

Since $a(b+c) > ab$ and $(a+b)b > ab$, therefore a product changes if either of its factors changes. A product increases if either of its factors increases.

Notwithstanding the historical origin of addition from counting and of multiplication from the addition of equal terms, it is now advantageous to consider multiplication, not as repeated addition, but as a separate operation, only connected with addition by the distributive law, an operation for finding from two elements x, y, an element univocally determined, xy, called "the product, x by y," which by commutativity equals x times y.

THE TWO INVERSE OPERATIONS, SUBTRACTION AND DIVISION

Inversion.

In the preceding direct operations, in addition and multiplication, the simplest problem is, from two given numbers to make a third.

If a and b are the given numbers, and x the unknown number resulting, then

$$x = a + b, \text{ or}$$
$$x = a \times b,$$

according to the operation.

An *inverse* of such a problem is where the result of a direct operation is given and one of the components, to find the other component. The operation by which such a problem is solved is called an inverse operation.

Since by the commutative law we are free to interchange the two parts or terms of a given sum, as also the two factors of a given product, therefore here the inverse operation does not depend upon which of the two components is also given, but only upon the direct operation by which they were combined.

Subtraction.

Suppose we are given a sum which we designate by s, and one part of it, say p, to find the corresponding other part, which, yet unknown, we represent by x. Since the sum of the numbers p and x is what $p+x$ expresses, we have the equality $x+p = s$.

But this equation differs in kind from the literal equalities heretofore used. It is not a formula, for any digital number substituted for one of these letters restricts the simultaneous values permissible for the others. Such an equality is called a conditional equality or a *synthetic* equation, or simply an *equation*.

The inverse problem for addition now consists just in this,—to solve the synthetic equation

$$b+x = a,$$

when a and b are given; in other words, to find a definite number which placed as value for x will satisfy the equation, that is which added to b will give a, and thus *verify* the equation. The number found, which satisfies the equation is called a *root* of the equation.

If the operation by which from a given sum a and a given part of it b we find a value for the corresponding other part x is called *from a subtracting b*, then, using the minus sign $(-)$ to denote subtraction, we may write the result $a-b$, read a minus b.

We may get this result, remembering that a number is a sum of units, by pairing off every unit in b with a unit in a, and then counting the unpaired units. This gives a number which added to b makes a.

The expression or result $a-b$ is called a *difference*.

The term preceded by the minus sign is called the *subtrahend;* the other, the *minuend*.

Thus $(a-b)+b = a-b+b = a$; also

$$b+(a-b) = b+a-b = a.$$

Ordinally, to subtract y from x is to find the number occupying the yth place before x.

Division.

The term division has two distinct meanings in elementary mathematics. There are two operations called division: 1°, Remainder division; 2°, Multiplication's inverse.

1°, Given two numbers, $a > b$, a the *dividend*, and b the *divisor*, the aim of *remainder division* may be considered the putting of a under the form $bq+r$, where $r < b$, and b not 0. We call q the *quotient*, and r the *remainder*. Both are integral.

The remainder division of a by b answers the two questions: 1°, What multiple of b if subtracted from a gives a difference or remainder less than b? 2°, What is this remainder?

When r is 0, then a is a *multiple* of b, and a is *exactly divisible* by b.

The case $b=0$ is excluded. In this excluded case the problem would be impossible if a were not 0. But if $a=0$ and $b=0$, every number, q, would satisfy the equality $a=bq$. So this case must be excluded to make the operation of division unequivocal, that is, in order that the problem of division shall have always one and only one solution. A second solution q', r' would give $a= bq+r=bq'+r'$, $b(q-q')=r'-r$. But $r'-r<b$, while $b(q-q')$ not $<b$.

2°, Division may also be regarded as the inverse of multiplication. Its aim is then considered to be the finding of a number q (quotient) which multiplied by b (the divisor) gives a (the dividend). Here division is the process of finding one of two factors when their product and the other factor are given.

The result q is represented by a/b. If $a=0$, then $q=0$. This definition of division gives the equality $(a/b)b=a$.

Remember $b \neq 0$, that is, b not equal to 0.

In particular $\qquad a/1=a.$

In general
1°, $(a+b)/m=a/m+b/m.$
2° $(a-b)/m=a/m-b/m.$
3° $a(b/c)=ab/c.$
4° $a/(bc)=(a/b)/c.$
5° $a/(b/c)=(a/b)c.$
6° $a/b=am/bm.$
7° $a/b=(a/m)/(b/m).$

TECHNIC.

Addition.

In adding a column of digits, consider two numbers together, but only *think* their sum.

	38
3	23
8	48
5	35
9	59
2	62
7	87
4	74
5	95
43	3

Now in adding up this column only think 9, 16, 18, 27, 32, 43, stressing forty, and writing down the three while thinking it.

The stress on the forty is to hold the four in mind for use in the next column to the left. Such a number is said to be *carried*. Begin adding up the next column to the left by thinking 13.

To check the work, add the column downward, since mere repetition of work tends to repeat the mistake also.

Subtraction.

Look at the question of subtracting as asking what number added to the subtrahend gives the minuend. Always work subtraction by adding. Thus subtract 1978 from 3139 as follows: Think, 8 and one make 9; 7 and six make 13, carry 1; 10 and one make 11, carry 1; 2 and one make 3. Write down the spelled digits just while thinking them.

```
 3139
 1978
 ----
 1161
```

Explain "carrying" by the principle that the difference between two numbers remains the same though they be given equal increments.

```
 9254
 8365
 ----
  889
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Again think, 5 and nine make 14, carry 1; 7 and eight make 15, carry 1; 4 and eight make 12, carry 1; 9 equals 9.

In working the examples we have added *downwards*, so check by adding *upwards* the difference (the answer) to the subtrahend, think (for 9 and 5) 14, (for 9 and 6) 15, (for 9 and 3) 12, (for 1 and 8) 9.

Multiplication.

Set down the multiplier precisely in column under the multiplicand, units under units. Begin by multiplying the units figure of the multiplicand by the leftmost figure of the multiplier, writing under this leftmost figure the first figure thus obtained. Then use the successive figures in order. The figure set down from multiplying the units always comes precisely under its multiplier.

```
    35427
     1324
 --------
 35427
 106281
   70854
   141708
 --------
 46905348
```

The advantage of this method is that it gives the most important partial product first, and in abridged or approximate work one or two of the leftmost figures may be all that are wanted.

Rule: If of two figures multiplied one is in units column, the figure set down stands under the other.

Verify Multiplication by Casting out Nines.

Proceed as follows: Add the single figures of the *multiplicand,* but always diminish the partial sums by dropping nine. The remainder is identical with the remainder found much more laboriously by dividing by nine. Thus 35427 gives 3, since 7 and 2 give nine as also 4 and 5. Find just so the remainder of the *multiplier.* Here 1324 gives 1. If our work is correct, the remainder, or *excess,* of the product of these two remainders equals the remainder, or excess, for our product. Here 46905348 gives 3.

The complete proof of this method of verification lies simply in the fact that the remainder left when any number is divided by nine is the same as that left when the sum of its digits is divided by nine. For 10–1=9, 100–1=99, 1000–1=999, etc. Hence if from any number be taken its units, also a unit for each of its tens, a unit for each of its hundreds, a unit for each of its thousands, etc., the remainder is a multiple of nine. But the part taken away is the sum of the number's digits.

Shorter Forms.

When the multiplier contains only two digits, shorten the work by adding in the results of the multiplication by the second digit to that already obtained.

```
  9587
    32
 -----
 28761
------
306784
```

Here, after multiplying by 3, think, four*teen*; 16, 17, eigh*teen*; 10, 11, seven*teen*; 18, 19, *twenty*-six; ten; three. Write down the unaccented part of these spelled numbers while thinking it.

If in a multiplier of only two digits either is unity, write only the answer.

```
  9867
    15
------
148005
```

Here think *thirty*-five; 30, 33, *forty*; 40, 44, *fifty*; 45, 50, *fifty*-eight; fourteen.

```
  7968
    41
------
326688
```

Here think eight; 32, *thirty*-eight; 24, 27, *thirty*-six; 36, 39, *forty*-six; 28, thirty-two.

When in a three-place multiplier taking away either end-digit leaves a multiple of it, shorten by adding to this digit's partial product the proper multiple of it.

```
  1234
   568
------
  9872
 69104
------
700912
```

After multiplying by 8, multiply this partial product by 7 (tens).

```
   4213
    864
-------
 33704
 269632
-------
3640032
```

After multiplying by the 8, (hundreds), multiply this partial product by 8. This gives units.

Division.

Write the first figure of the quotient precisely over the last figure of the first partial dividend. Use no bar to separate them.

Omit the partial products, the multiples of the divisor, writing down the partial dividends, the differences, while doing the multiplication.

27	Here think 16 and nought, 1'6. Carry 1, 10,
358) 9762	11 and six, 1'7. Carry 1. 6, 7 and two, 9. 56 and
260	six, 6'2. Carry 6. 35, 41 and nine, 5'0. Carry 5.
96	21, 26.

Verify Division by Casting Out Nines.

The excess of the product of excesses of divisor and quotient increased by excess of remainder equals excess of dividend.

In our example the excess from the quotient is 0. So the excess from the dividend, 6, equals that from the remainder.

[TO BE CONTINUED.]